Wireless Sensor Networks Localization with Outliers and Structured Missing Data

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***Abstract*—In this paper we address the Wireless Sensor Networks localization problem in a realistic scenario with outliers and *structured* missing data (i.e. non-random). Our approach couples SMACOF, which handles incomplete data, with IRLS, which is resilient to outliers. In addition, we provide a new insight on how the initialization method – which is crucial to ensure fold-free solutions – should be adapted to the pattern of missing measures. Experiments shows that the proposed method compares favorably with the state-of-the-art.**

I. Introduction

Wireless Sensor Networks (WSNs) are collections of autonomous electronic devices spread over an indoor or outdoor area that are able to record physical data and cooperate over a wireless link. Typically, the network is *ad-hoc*, and sensors can be fixed or mobile. Each node consists – at least – of a low power processor, a limited amount of memory, a sensor board and a wireless network transceiver.

WSNs are usually deployed in several applications, including environmental monitoring, failure detection or reporting in smart buildings, and target tracking. In such applications it is necessary to orient the nodes with respect to a global coordinate system in order to express all the measurements in a common reference frame, and report data that is geographically meaningful. Additionally, basic network services such as routing often rely on location information.

Instead of using globally accessible beacons or GPS to localize each sensor, Sensor Network Localization (SNL) requires the sensors to *self-organize* a coordinate system. Only a few nodes, known as *anchor nodes*, have prior knowledge about their coordinates, either hard coded, or acquired through some additional hardware like a GPS receiver, which is economically expensive and energy-consuming. All the other nodes must determine their position by measuring *range* or *bearing* with respect to other nodes. This work focuses on the case of range measurements.

Distances between pairs of nodes can be measured using the Received Signal Strength Indicator (RSSI) of radio signals sent by neighbor sensors. Indeed, the energy of a radio signal diminishes with the distance from the signal's source. As a result, a node listening to a radio transmission should be able to use the strength of the received signal to estimate its distance from the transmitter. Likewise, sensors nodes can asses the Time of Arrival (TOA) or the Time Difference of Arrival (TDOA) of radio or acoustic signals emitted by neighbor sensors, which in turn can be converted into pairwise distance estimates.

Measuring distances using RSSI or ToA is subject to noise, rogue measures (or outliers), and missing data. Noise depends on radio propagation which tends to be highly non-uniform, outliers are due to environmental factors (reflections and interferences) or hardware malfunctioning (suffering from transmitter, receiver, and antenna variability), and missing data depend on limitations in the radio range or shields.

From the theoretical point of view, determining the positions of nodes from their distances is formalized as a Multidimensional Scaling (MDS) problem, which can be solved easily when all the distances have been measured and there are no outliers. Missing data have been overcame by the SMACOF (Scaling by MAjorizing a COmplicated Function) formulation [1], [2], while outliers have been addressed by several robust techniques, including replacing the ℓ_2 norm with ℓ_1, (Semidefinite Programming (SDP) [3], Robust MDS (RMDS) [4]) and low-rank and sparse matrix factorization (ER-GoDec [5]).

SDP [3] is a robust technique that copes with unspecified measurements. The idea consists in casting the SNL problem to a standard SDP, which is accurate but has a high computational complexity.

The ER-GoDec solution leverages on recent advances in the field of Low-Rank and Sparse matrix decompositions to complete the range information between pairs of sensors in the presence of rogue measures. In particular, the authors designed a cost function that not only includes unspecified measurements and outliers in its definition, but also constrains the solution to stay close to an Euclidean Distance Matrix (EDM), i.e., a matrix containing squared distances between nodes.

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RMDS [4] is a SMACOF-like algorithm that introduces a ℓ_1-based regularization term into the cost function in order to manage outliers. More details on the most relevant methods to this work will be given in Sec. II

In this paper we propose to use Iteratively Reweighted Least Squares (IRLS) [6] coupled with SMACOF to obtain a localization which is resilient to outliers and handles incomplete measures; the method will be detailed in Sec. III. Experiments, reported in Sec. IV, concentrate on our closest competitor [4] and report better performances in accuracy with comparable computation time.

In addition, we provide a new insight on the initialization required for different pattern of missing data, which is crucial in making a methods to succeed. For example, RMDS [4] was able to produce good results only with random missing measures because of the specific initialization.

II. Theoretical Background

A sensor network in a m-dimensional space (in our case $m = 2$) consists of n nodes, and a set of e Euclidean distances between pairs of nodes. The network can be described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ represents sensors and the edge set $\mathcal{E}$ of cardinality e is defined by pairs (i, j) of sensors for which a distance measurement is available. $\mathcal{G}$ is called *distance graph*.

The coordinates of the nodes are described by n points in $\mathbb{R}^m$ which can be ascribed to the rows of a matrix $X = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]^T \in \mathbb{R}^{n\times m}$. X is usually referred to as a *configuration* or *embedding*. We denote as $D(X)$ the matrix of distances between pairs of nodes, i.e.

$$\begin{aligned} D_{ij} = d_{ij} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \quad \text{if } (i,j) \in \mathcal{E} \\ d_{ij} = 0 \quad \text{if } (i,j) \notin \mathcal{E} \end{aligned} \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$.

In real applications measured distances could be affected by noise and outlier measurements. Given the Euclidean distance function $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined over pairs of nodes, we model the measures as

$$\widehat{d}_{ij} = d_{ij} + o_{ij} + \epsilon_{ij} \quad \forall (i,j) \in \mathcal{E} \tag{2}$$

where o_{ij} represents an outlier component and ϵ_{ij} an additive noise (we assume i.i.d. $\epsilon_{ij} \sim \mathcal{N}(0, \sigma^2) \ \ \forall (i,j) \in \mathcal{E}$). We denote as $\widehat{D}$ the matrix of *measured* distances, henceforth called *data matrix*.

The SNL problem consists in recovering a configuration X given $\widehat{D}$. Equivalently, SNL can be seen as a *graph realization problem*, where the goal is to assign coordinates $\boldsymbol{x}_i \in \mathbb{R}^m$ to each node so that the Euclidean distances between pairs of nodes match (or *realize*) the distances of the corresponding edges.

A sensor network is *uniquely localizable* if exists a unique X that realizes the distances in $\widehat{D}$ (up to a rigid transformation). It is know that if the distance graph $\mathcal{G}$ is *generically globally rigid* (GGR for short) then the network is uniquely localizable [7]. In two dimensions the distance graph is GGR if and only if it is *3-connected* and *redundantly rigid* [8]. In particular, these two properties can be verified in polynomial time [9], [10].

One of the most famous class of methods which solves the SNL is MDS [11]. MDS, originally proposed as a visualization tool in data analysis [12], recovers an embedding in $\mathbb{R}^m$ such that pairwise distances approximate the given data matrix.

Two simple methods of this class are classical MDS [12] and SMACOF [1]. Classical MDS is effective only when all pairwise distance measurements are available (i.e. $\widehat{D}$ is complete), instead SMACOF can manage unspecified distances. Despite they perform well with exact distances, both classical MDS and SMACOF are sensitive to outliers and yield roughly incorrect embeddings even if few outliers corrupt the data matrix.

In order to handle both unspecified data and outliers, a robust MDS formulation (called RMDS) has been proposed in [4] which is based on a distance model that explicitly accounts for outliers. We now review these MDS based localization techniques more in detail.

A. Classical MDS

Classical MDS minimizes the following cost function

$$\min_X \left\| -\tfrac{1}{2} J \widehat{D}^2 J - X X^T \right\|_F^2 \tag{3}$$

where $J = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ is the *centering operator*, I is the identity matrix, $\mathbf{1}$ is the column vector of all ones, and $\|\cdot\|_F$ denotes the Frobenius norm. The method (summarized in Algorithm 1) computes the embedding as the m-first principal components of $-\frac{1}{2} J \widehat{D}^2 J$. Classical MDS does not cope with unspecified measurements and outliers, thus its workability is limited in practice.

Algorithm 1: Classical MDS

Input : $\widehat{D}$, m
Output: X

$J = I - \mathbf{1}\mathbf{1}^T/n$;
$G = -\frac{1}{2} J \widehat{D}^2 J$;
$[Q, \Lambda] = \mathrm{eig}(G)$;
$X = Q^T \Lambda_m^{1/2}$;

B. SMACOF

The SMACOF approach [1], [2], solves the SNL problem through a majorization technique also in the event that $\widehat{D}$ is not complete. SMACOF minimizes the following *stress function*

$$\sigma(X) \doteq \sum_{i,j=1}^{n} \omega_{ij} (\widehat{d}_{ij} - d_{ij})^2 \tag{4}$$

where $W = [\omega_{ij}]_{i,j=1}^{n} \in \mathbb{S}_h$ is a matrix of weights which equals the *adjacency matrix* of the distance graph (namely its entries are equal to 1 if $\widehat{d}_{ij}$ is specified, and 0 otherwise). Observe that missing measurements are accounted in the SMACOF framework thanks to the introduction of the matrix W. SMACOF finds a *majorizing function* of $\sigma(X)$ (i.e. a function $\tau(X, Y)$ such that $\sigma(X) \leq \tau(X, Y)$ for all $Y \in \mathbb{R}^{n \times m}$ and $\sigma(X) = \tau(X, X)$) and attempts to minimize it.

To this aim the stress function is decomposed in the following way

$$\sigma(X) = \|\widehat{D}\|_F + \sum_{i,j=1}^{n} d_{ij}^2 - 2 \sum_{i,j=1}^{n} \omega_{ij} \widehat{d}_{ij} d_{ij}. \quad (5)$$

Denoted by L the *Laplacian* of the distance graph (in this case $L = \operatorname{diag}(W\mathbf{1}) - W$), the second term of the right member of Eq. (5) can be rewritten as $\sum_{i,j=1}^{n} d_{ij}^2 = \operatorname{tr}(X^T L X)$ where $\operatorname{tr}(\cdot)$ is the trace operator. Consider now the following matrix

$$B(X) \doteq \operatorname{diag}(A(X)\mathbf{1}) - A(X) \quad (6)$$

where

$$[A(X)]_{ij} \doteq \begin{cases} \omega_{ij} \frac{\widehat{d}_{ij}}{d_{ij}} & \text{if } \widehat{d}_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Observe that $A(X)$ is the adjacency matrix of a weighted graph $\mathcal{G}_1 = \{\mathcal{V}, \mathcal{E}_1\}$, where edge weights are defined by the entries of $A(X)$, and that $B(X)$ is the corresponding Laplacian matrix (or *weighted Laplacian*). Then the third term of the right member of Eq. (5) can be expressed as $-2 \sum_{i,j=1}^{n} \omega_{ij} \widehat{d}_{ij} d_{ij} = -2 \operatorname{tr}(X^T B(X) X)$. Consequently, the stress decomposition becomes

$$\sigma(X) = \|\widehat{D}\|_F + \operatorname{tr}(X^T L X) - 2 \operatorname{tr}(X^T B(X) X). \quad (8)$$

Let now $Y \in \mathbb{R}^{n \times m}$ be another configuration. By an application of the Cauchy-Schwartz inequality, it follows that for all pairs of configuration $X, Y \in \mathbb{R}^{n \times m}$ holds

$$\begin{aligned} \sigma(X) &\leq \|\widehat{D}\|_F + \operatorname{tr}(X^T L X) - 2 \operatorname{tr}(X^T B(Y) Y) \\ &\doteq \tau(X, Y). \end{aligned} \quad (9)$$

In particular, τ is a (simple) quadratic function in X and Y which majorizes the stress σ. A closed-form solution for the minimum is

$$X = L^+ B(Y) Y \quad (10)$$

where L^+ is the Moore-Penrose inverse of L

$$L^+ \doteq (L + n^{-1}\mathbf{1}\mathbf{1}^T)^{-1} - n^{-1}\mathbf{1}\mathbf{1}^T. \quad (11)$$

The SMACOF method, summarized in Algorithm 2, is efficient, but not robust to outliers. Moreover, it needs to be properly initialized. Especially when the pattern of $\widehat{D}$ has some structure, SMACOF tends to find local minima because of the *foldover* phenomena, thus the challenge is to generate an initial solution which is *fold-free*. The issue of initialization will be discussed in Sec. III-B

Algorithm 2: SMACOF

Input : X^0, $\widehat{D}$, tol
Output: X

```
t = 0;
compute L^+;
while σ(X^t) > tol do
    compute B(X^t);
    X^{t+1} = L^+ B(X^t) X^t;
    t = t + 1;
end
```

C. Robust MDS

Forero et al. [4] introduced a cost function which takes outliers into account. Then, in a similar way of SMACOF, they derive a majorizing of the cost function and find out closed-form updates for solving the minimization. The cost function is defined as follows

$$\sigma(O, X) \doteq \sum_{i,j=1}^{n} \omega_{ij} (\widehat{d}_{ij} - d_{ij} - o_{ij})^2 + \lambda \sum_{i,j=1}^{n} \omega_{ij} |o_{ij}| \quad (12)$$

where $O = [o_{ij}]_{i,j=1}^{n} \in \mathbb{S}_h$ is an unknown outlier matrix. Observe that the second ℓ_1-norm term is a regularization term that controls the *sparsity* of the matrix O.

With a reasoning that mimics that of SMACOF, it is possible to derive a majorizer $\tau(O, X, Y)$ for $\sigma(O, X)$ (i.e. such that $\tau(O, X, Y) \geq \sigma(O, X)$ for all $O \in \mathbb{R}^{n \times n}$ and $X, Y \in \mathbb{R}^{n \times m}$, and $\tau(O, X, X) = \sigma(O, X)$). Variables (O, X) are then updated through the following alternate minimization

$$O^{t+1} = \operatorname*{argmin}_{O} \tau(O, X^t, X^t) = \operatorname*{argmin}_{O} \sigma(O, X^t) \quad (13)$$

$$X^{t+1} = \operatorname*{argmin}_{X} \tau(O^{t+1}, X, X^t). \quad (14)$$

The solution of (13) can be expressed in terms of the soft-thresholding operator $\Theta_\lambda(x) = \operatorname{sign}(x) \circ \max\{0, |x| - \lambda\}$ as

$$O^{t+1} = \Theta_\lambda(\widehat{D} - D(X^t)) \quad (15)$$

where scalar operations are applied element-wise. While the solution of (14) is

$$X^{t+1} = L^+ B_1(O^{t+1}, X^t) X^t \quad (16)$$

where

$$B_1(O, X) \doteq \operatorname{diag}(A_1(O, X)\mathbf{1}) - A_1(O, X) \quad (17)$$

and

$$[A_1(O, X)]_{ij} \doteq \begin{cases} \omega_{ij} \frac{(\widehat{d}_{ij} - o_{ij})}{d_{ij}} & \text{if } \widehat{d}_{ij} > o_{ij},\ d_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

Observe that $A_1(X)$ is the adjacency matrix of a weighted graph $\mathcal{G}_2(\mathcal{V}, \mathcal{E}_2)$, where edge weights are defined by the non-zero entries of $A_1(X)$ and $B_1(X)$ is the corresponding Laplacian. On the whole, Eq. (14) is the same of Eq. (10) excepts for $B_1(O^{t+1}, X^t)$ instead of $B(X^t)$. A more detailed discussion about RMDS can be found in [4]. The method is summarized in Algorithm 3.

The main drawback of RMDS is that in general the soft-thresholding parameter λ needs to be estimated. In particular, the optimal value can be found *a posteriori* through the true localization knowledge or having some more information about outliers sparsity. Usually a suitable value of λ is chosen via experimental analysis.

Algorithm 3: RMDS

Input : X^0, $\widehat{D}$, tol
Output: X

$t = 0$;
$W = \text{logical}(\widehat{D})$;
compute L^+;
$O^0 = \text{zeros}(n)$;
while $\sigma(O^t, X^t) > tol$ **do**
  $O^{t+1} = \Theta_\lambda(W \circ (\widehat{D} - D(X^t)))$;
  compute $B_1(O^{t+1}, X^t)$;
  $X^{t+1} = L^+ B_1(O^{t+1}, X^t) X^t$;
  $t = t + 1$;
end

III. Proposed Method

Our solution to the SNL problem combines the SMACOF approach with the Iteratively Reweighted Least-Squares (IRLS) algorithm [6]. We start from the least-squares minimization (4) and we embed it into an IRLS framework in order to handle rogue measurements. This technique iteratively solves weighted least squares problems where the weights are estimated at each iteration by analyzing the residuals of the current solution. The weights are assigned by a specific weight function in such a way to penalize outliers and promote inliers. We dubbed this novel approach SMACOF-IRLS.

A. SMACOF-IRLS

We propose to embed the SMACOF method into an IRLS scheme in the following way. We first obtain an estimate of X solving (4) using Algorithm 2. In this first step weights are defined by the adjacency matrix of the distance graph (i.e. they are all 1 or 0 depending if pairwise distance measurements are available or not). Then we update the weights ω_{ij} using the current estimate of the embedding and the Cauchy weight function:

$$\omega_{ij} = \frac{1}{1 + \left(\frac{r_{ij}}{c}\right)^2} \tag{19}$$

with $r_{ij} = \widehat{d}_{ij} - d_{ij}(X)$ if $\widehat{d}_{ij}$ is defined, 0 otherwise. The scaling factor c is computed as customary with

$$c = 2.385\sigma \tag{20}$$

where $\sigma = \text{med}(\boldsymbol{r} - \text{med}(\boldsymbol{r}))/0.6745$, $\text{med}(\cdot)$ is the median of a vector, and $\boldsymbol{r}$ is the vectorization of the residuals r_{ij}.

In matrix form, at each step W is updated using the residual matrix $R = \widehat{D} - D(X)$ through Eq. (19). Observe that updating W results in a updated adjacency matrix $A(X)$ in Eq. (7), and consequently in an updated weighted Laplacian $B(X)$. Then a new step of SMACOF is applied with the updated variables. These steps are iterated until convergence or a maximum number of iterations is reached. The SMACOF-IRLS method is summarized in Algorithm 4.

We verified that convergence is improved by starting with few iterations and progressively increasing them after each weight update. The rationale is that it does not make sense to refine too much the SMACOF solution at the beginning, when outliers identification is still unsure. In this way, at the beginning weights are updated frequently so that the IRLS contribute prevails and outliers starts to emerge. Then, as IRLS stabilizes, the number of iterations of SMACOF (denoted as *maxIterIn* in the pseudocode) are increased, in order to improve the final accuracy. In particular, at the k-th step of IRLS we fix *maxIterIn* $= \lceil \alpha^{k-1} \rceil$ with $\alpha > 1$.

Algorithm 4: SMACOF-IRLS

Input : X^0, $\widehat{D}$, *maxIterOut*, α. tol
Output: X

$W^0 = \text{logical}(\widehat{D})$;
for $k = 1 :$ *maxIterOut* $+ 1$ **do**
  $t = 0$;
  compute L^+;
  maxIterIn $= \lceil \alpha^{k-1} \rceil$;
  while $\sigma(X^t) > tol \vee t <$ *maxIterIn* **do**
    compute $B(X^t)$;
    $X^{t+1} = L^+ B(X^t) X^t$;
    $t = t + 1$;
  end
  $R^k = W^0 \circ (\widehat{D} - D(X^t))$;
  compute c;
  $W^k = W^0 \circ (1 + (R^k/c)^2)^{-1}$;
end

B. Initialization

SMACOF, RMDS, and IRLS-SMACOF share the same iterative scheme, which requires a proper initialization X^0 in order to converge to the global minimum.

One possibility is to use classical MDS to derive X^0, as suggested in [4]. Indeed, even if classical MDS does not provide a good solution for SNL in the event that

$\widehat{D}$ is not complete, it can be a suitable starting point for RMDS (and SMACOF as well), under certain hypothesis to be further specified.

Another possible initialization is the so-called *Laplacian Eigenmap* proposed in [2]. The following energy function is considered

$$E(X) = \frac{\sum_{i,j} \omega_{ij} d_{ij}^2}{\sum_{i,j} d_{ij}^2} = \frac{\|W \circ D(X)\|_F^2}{\|D(X)\|_F^2} \qquad (21)$$

where $\circ$ is the Hadamard product. Examples of possible weights are $\omega_{ij} = 1/(\widehat{d}_{ij} + \beta)$ or $\omega_{ij} = \exp(-\widehat{d}_{ij}/\beta)$ with ($\beta \geq 0$) if d_{ij} is available, and 0 otherwise. The global minimum of $E(X)$ is given by the eigenvectors of the related weighted Laplacian $\mathrm{diag}(W\mathbf{1}) - W$. In practice, as explained in [13], it is better to work with the closely related eigenvectors of the *transition matrix* $\mathrm{diag}(W\mathbf{1})W$. Note that the top eigenvalue of the transition matrix is $\lambda_1 = 1$, which is associated with the constant eigenvector $\boldsymbol{v}_1 = \mathbf{1}$, so the embedding is actually given by the eigenvectors from the second to the $m+1$-th.

Our experiments have shown that the initialization methods has to be chosen according to the pattern of unspecified entries. In particular, MDS is a good choice if the pattern of unspecified entries is random, whereas Laplacian Eigenmap do not work effectively in this case. On the other hand, when unspecified data follow a more realistic pattern that depends on the distance (namely, where all measures greater than a fixed distance are missing), the situation switches: initialization with MDS fails, whereas Laplacian Eigenmap is effective.

An explanation of these facts follows from the minimization of (21) solved by Laplacian Eigenmap. Indeed minimizing $E(X)$ leads to locate adjacent sensors close one to each other while separating nonadjacent sensors. More precisely, since the unspecified entries appear in the denominator of $E(X)$ only, then during the minimization they tend to become greater than the specified ones. Thus Laplacian Eigenmap respects range-based patterns since the unspecified entries are actually greater than the specified ones, while it is ineffectual for random patterns since the relationship between unspecified and specified entries is no longer valid. On the other side, classical MDS cannot manage range-based patterns since it equally weights each measurement without taking into account the structure given by the range. For the same reason, it is a good choice for random patterns, instead.

IV. Experimental Results

We provide several synthetic experiments in MATLAB® in order to validate our solution. We compared SMACOF-IRLS against three robust approaches, namely ER-GoDec, RMDS, and SDP, the latter being considered the state-of-the-art.

The implementations of SDP (based on the SDPT3 solver [14]) and of RMDS [4] are available online, while for ER-GoDec and SMACOF-IRLS we use our implementations. From preliminary analysis we fixed $\alpha = 1.3$ and *maxIterOut* $= 21$ in Algorithm 4 and $\lambda = 1.25$ in Algorithm 3. The same initialization is provided to all algorithms.

We compare the performances of the methods in terms of i) noise resilience and robustness to outliers, ii) sensitivity to missing data (both random and structured), and iii) efficiency.

In each experiment, we randomly generate n points in a square of side l centered in the origin, representing the ground truth node positions. We perturb the true pairwise distances with Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma^2)$. We discard a portion ($H\%$) of distances or all the measure greater than a fixed range (R) to simulate unspecified data. We set a portion ($O\%$) of the specified distances to random values in the interval $[0, l]$ to simulate outliers. The localizability of the distance graph is checked using our implementations of redundant rigidity [9] and 3-connectivity [10]. Results are averaged over 100 trials, discarding the simulations in which the localizability check fails. As figure of merit we used the *Root Mean Square Error* (RMSE)

$$RMSE = \sqrt{\tfrac{1}{n} \left\| X - \tilde{X} \right\|_F^2} \qquad (22)$$

where $\tilde{X}$ is the estimated configuration and X the true one. The RMSE is computed after having aligned $\tilde{X}$ onto $\tilde{X}$ using Orthogonal Procrustes Analysis [15].

A. Noise Resilience and Robustness to Outliers

In this experiment we first evaluate the noise resilience of our method SMACOF-IRLS without considering missing and outliers data. We set $n = 100$, $l = 100$, $H = 0\%$ (no undefined entries), and $O = 0\%$ (no outliers). In this case the localizability test is trivial. Results are reported in Fig. 1.

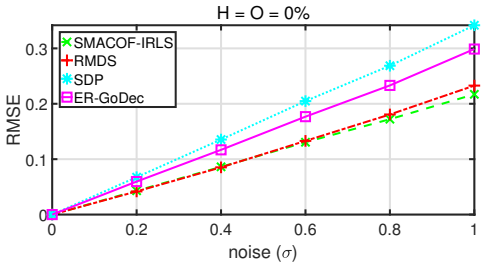

Figure 1. Noise Resilience. On the x-axis the noise std deviation σ, on the y-axis the corresponding (average) RMSE obtained with our method SMACOF-IRLS and competitors. The number of nodes is $n = 100$ and the side of the square is $l = 100$. Nor missing nor outlier data are present ($H = O = 0\%$).

Observe that MDS based localization techniques (SMACOF-IRLS and RMDS) achieve the most accurate results, followed by ER-GoDec and SDP. This depends on the fact that the SMACOF maximizes a likelihood

function[16], hence it is statistically optimal in presence of i.i.d. Gaussian noise.

Since all errors grow linearly with respect to σ, we decided to fix a reasonable level of noise, namely $\sigma = 0.6$, for the following experiments. This level of noise provides an RMSE approximately lower than 0.2 for all the methods.

We then evaluate the robustness to outliers of SMACOF-IRLS without considering missing data. The parameter setup is $n = 100$, $l = 100$, $\sigma = 0.6$, and $H = 0\%$. Results are shown in Fig. 2.

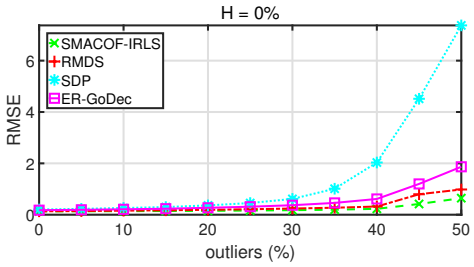

Figure 2. Robustness to Outliers. On the x-axis the portion of outliers, on the y-axis the corresponding (average) RMSE obtained with our method SMACOF-IRLS and competitors. The number of nodes is $n = 100$, the side of the square is $l = 100$, the noise std deviation is $\sigma = 0.6$. No missing data are present ($H = 0\%$).

SMACOF-IRLS is robust to rogue measurements since it produces good results up to a fraction of 40% outliers. RMDS and ER-GoDec show a behavior similar to SMACOF-IRLS with slightly worse results for portions of outliers greater than the 40%. SDP has the worst breaking point at 30%. On the whole, all methods can be considered robust to outliers when the data matrix is complete.

B. Sensitivity to Unspecified Data

In the following two experiments we study the sensitivity of SMACOF-IRLS to unspecified data. We analyze two different patterns of unspecified entries, namely random and range-based (i.e. where all measures greater than a fixed distance R are missing). MDS based localization techniques have been initialized according the criteria explained in Section III-B.

1) Random: We consider random patterns in presence of outliers. We set $n = 100$, $l = 100$, $\sigma = 0.6$, and we analyze the behaviors of the methods respectively with respect to unspecified entries (fixing $O = 30\%$) and to the outliers (fixing $H = 30\%$). Results are summarized in Fig. 3. The upper plot shows that up to 60% of unspecified entries all methods achieve comparable results proving their non-sensitivity to random missing data. When H is greater than 70% all the methods worsen. The lower plot shows that ER-GoDec and SMACOF-IRLS achieve the most accurate results when both missing data and outliers are present. RMDS and SDP obtain slightly worse results. As for localizability,

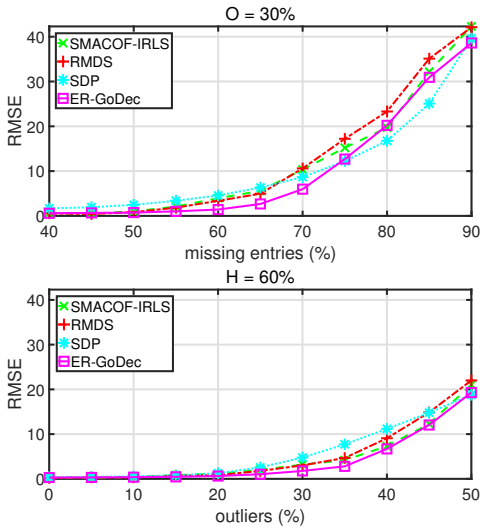

Figure 3. Sensitivity to Random Unspecified Data. On the x-axis the portion of random unspecified data (upper plot) and the portion of outliers (lower plot), on the y-axis the corresponding (average) RMSE obtained with our method SMACOF-IRLS and competitors. The number of nodes is $n = 100$, the side of the square is $l = 100$, the noise std deviation is $\sigma = 0.6$. In the upper plot $O = 30\%$, while in the lower plot $H = 60\%$.

the check failed 17 times over 100 simulations when the portion of unspecified entries was 90% (rightmost point in the upper plot), while for lower portions the check never failed.

2) Range-Based: This situation is more realistic, since in a real WSN each node can communicate only with its closest neighbors. We set $n = 100$, $l = 100$, $\sigma = 0.6$, and we analyze the behaviors of the methods respectively with respect to the range (fixing $O = 30\%$) and to the outliers (fixing $R = 50$, which corresponds to about 52% of unspecified entries). Results are summarized in Fig. 4.

The upper plot shows that SMACOF-IRLS, RMDS and SDP are not sensitive to range-based unspecified entries for ranges greater than 40 (i.e. with approximatively 65% of unspecified entries), where all three methods have the breaking point. ER-GoDec has the worst breaking point at $R = 80$ (i.e. with approximatively 15% of unspecified entries). The lower plot confirms the bad performance of ER-GoDec for all portions of outliers. SMACOF-IRLS obtains the best accuracy followed by RMDS up to a portion of outliers of 45%, where it has its breaking point. SDP has its breaking point at about 30% of outliers.

The localizability check failed 82 times over 100 simulations when the range was 20 (leftmost point in the upper figure), which corresponds to approximatively 90% of unspecified entries, while for ranges greater than 20 the check never failed. Observe that graphs obtained by removing edges according to a range R are more likely non-localizable compared to graphs obtained

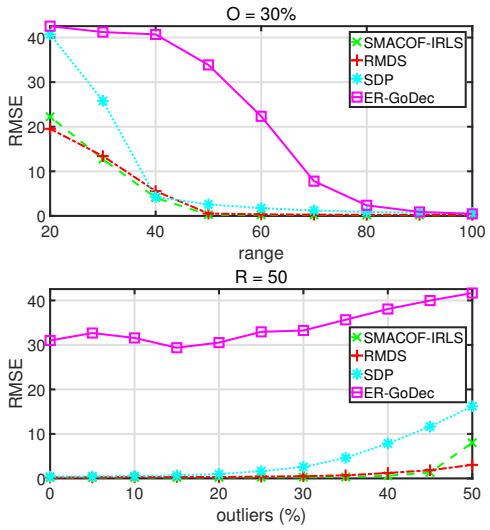

Figure 4. Sensitivity to Range-based Unspecified Data. On the x-axis the range (upper plot) and the portion of outliers (lower plot), on the y-axis the corresponding (average) RMSE obtained with our method SMACOF-IRLS and competitors. The number of nodes is $n = 100$, the side of the square is $l = 100$, the noise std deviation is $\sigma = 0.6$. In the upper plot $O = 30\%$, while in the lower plot the $R = 50$.

by removing random edges (considering commensurate values of R and H).

C. Efficiency

In this experiment we analyze the efficiency of our solution. We set $n = 100$, $l = 100$, $\sigma = 0.6$, $O = 10\%$, $H = 30\%$, and we evaluate the execution time as a function of the number of nodes. In this case the localizability never fails. Results are reported in Fig. 5.

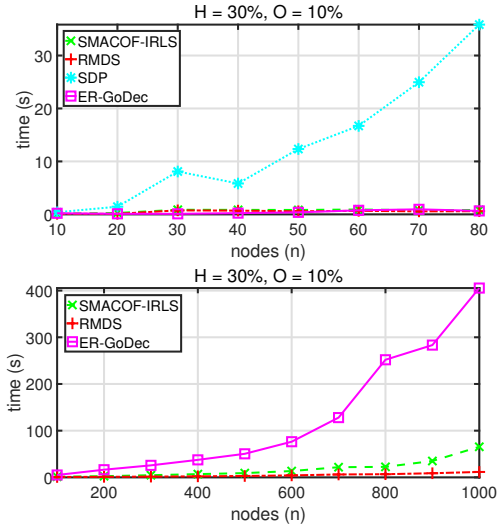

Figure 5. Efficiency. On the x-axis the number of nodes, on the y-axis the corresponding (average) execution time obtained with our method SMACOF-IRLS and competitors. The number of nodes is $n = 100$, the side of the square is $l = 100$, the noise std deviation is $\sigma = 0.6$, the portion of outliers is $O = 10\%$, and the portion of unspecified entries is $H = 30\%$.

The upper plot of Fig. 5 shows that up to 80 nodes SMACOF-IRLS, RMDS and ER-GoDec obtain comparable results, while SDP is definitively slow. Excluding SDP from the comparison, we increased the number of nodes up to 1000 as shown by the lower plot of Fig. 5. Among the remaining ER-GoDec is the slowest, while SMACOF-IRLS shows a behavior similar to RMDS which turns to be the most efficient.

V. Conclusions

We proposed a new method for WSN localization that couples IRLS and SMACOF to obtain resiliency to outliers and handling of incomplete measures. In addition, we found empirically that different patterns of missing data requires different initialization. In particular, when the pattern is not random but depends on the range, the Laplacian Eigenmap is the best choice. Experiments report better performances in accuracy with comparable computation time with respect to the closest competitors.

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